

In my paper [1] the definition of the notion of local isotropy was stated and the functions

$$B_{dd}(r) = \overline{[u_d(M') - u_d(M)]^2}, \quad B_{nn}(r) = \overline{[u_n(M') - u_n(M)]^2} \quad (1)$$

were introduced, where r denotes the distance between the points M and M' , $u_d(M')$ and $u_d(M)$ are the velocity components at the points M and M' in the direction of $\overline{MM'}$, and $u_n(M)$ and $u_n(M')$ are the velocity components at M and M' respectively, in a direction perpendicular to $\overline{MM'}$.

In what follows we also need the third order moments

$$B_{ddd}(r) = \overline{[u_d(M') - u_d(M)]^3}. \quad (2)$$

Locally isotropic turbulence in an incompressible fluid is described by the equation

$$4\bar{\epsilon} + \left(\frac{dB_{ddd}}{dr} + \frac{4}{r} B_{ddd} \right) = 6\nu \left(\frac{d^2 B_{dd}}{dr^2} + \frac{4}{r} \frac{dB_{dd}}{dr} \right), \quad (3)$$

which is analogous to the well-known von Karman equation for isotropic turbulence in Taylor's sense. Here $\bar{\epsilon}$ denotes the average energy dissipation rate per unit mass. Equation (3) can be written as

$$\left(\frac{d}{dr} + \frac{4}{r} \right) \left(6\nu \frac{dB_{dd}}{dr} - B_{ddd} \right) = 4\bar{\epsilon} \quad (4)$$

and, by virtue of the condition $dB_{dd}(r)/dr|_{r=0} = B_{ddd}(0) = 0$, it yields

$$6\nu dB_{dd}/dr - B_{ddd} = (4/5)\bar{\epsilon}r. \quad (5)$$

As is well known, for small r we have

$$B_{dd} \approx (1/15\nu)\bar{\epsilon}r^2, \quad (6)$$

that is, $6\nu dB_{dd}/dr \approx (4/5)\bar{\epsilon}r$.

Thus, for small r the second term on the left-hand side of equation (5) is infinitesimally small relative to the first term. Conversely, for large r the first term is negligibly small as compared to the second term, that is, we can assume that

$$B_{ddd} \approx -(4/5)\bar{\epsilon}r. \quad (7)$$

It is natural to assume that for large r the ratio

$$S = B_{ddd} : B_{dd}^{3/2}, \quad (8)$$

* Dokl. Akad. Nauk SSSR 32:1 (1941), 19-21.

that is, the *skewness* of the distribution of the difference

$$\Delta u_d = u_d(M') - u_d(M),$$

remains constant. Under this assumption, for large r we have

$$B_{dd} \approx C\bar{\epsilon}^{2/3}r^{2/3}, \quad (9)$$

where

$$C = (-4/5S)^{2/3}. \quad (10)$$

Relation (9) was derived in [1] on the basis of somewhat different considerations.¹

In [1] we introduced the *local scale* of turbulence

$$\lambda = (\nu^3/\bar{\epsilon})^{1/4} \quad (11)$$

and justified the assumption that

$$B_{dd}(r) = \sqrt{\nu\bar{\epsilon}}\beta_{dd}(r/\lambda), \quad B_{nn}(r) = \sqrt{\nu\bar{\epsilon}}\beta_{nn}(r/\lambda), \quad (12)$$

where β_{dd} and β_{nn} are universal functions such that

$$\beta_{dd}(\rho) \approx (1/15)\rho^2, \quad \beta_{nn}(\rho) \approx (2/15)\rho^2 \quad (13)$$

for small ρ and

$$\beta_{dd}(\rho) \approx C\rho^{2/3}, \quad \beta_{nn}(\rho) \approx (4/3)C\rho^{2/3} \quad (14)$$

for large ρ . In isotropic turbulence in Taylor's sense the laws of locally isotropic turbulence must hold at distances much less than the *integral scale* of turbulence L (for a precise definition see my paper [3]). In this case the correlation coefficients

$$R_{dd}(r) = \overline{(u_d(M')u_d(M))} : b, \quad R_{nn}(r) = \overline{(u_n(M')u_n(M))} : b, \quad (15)$$

where b is the average value of the square of the velocity components, are related to $B_{dd}(r)$ and $B_{nn}(r)$ by

$$B_{dd} = 2b(1 - R_{dd}), \quad B_{nn} = 2b(1 - R_{nn}). \quad (16)$$

By (16) and (12), for r small relative to L we must have

$$1 - R_{dd} \approx \frac{\sqrt{\nu\bar{\epsilon}}}{2b}\beta_{dd}\left(\frac{r}{\lambda}\right), \quad 1 - R_{nn} \approx \frac{\sqrt{\nu\bar{\epsilon}}}{2b}\beta_{nn}\left(\frac{r}{\lambda}\right). \quad (17)$$

¹ A.M. Obukhov obtained relation (9) independently, by calculating the balance in the spectral distribution of the fluctuation energy (see [2]).

If r is small relative to L and large relative to λ , then by (14) and (11),

$$1 - R_{dd} \approx (1/2)C\bar{\epsilon}^{2/3}b^{-1}r^{2/3}, \quad (18^1)$$

$$1 - R_{nn} \approx (2/3)C\bar{\epsilon}^{2/3}b^{-1}r^{2/3}. \quad (18^2)$$

Formulas (18) make it possible to determine the constant C from experimental data. The correlation coefficients R_{dd} and R_{nn} were measured most thoroughly by Dryden, Schubauer, Mock, and Scramstad [4]. Representing formula (18²) in the form

$$1 - R_{nn} \approx 2C(kr)^{2/3}, \quad k = \bar{\epsilon} : (3b)^{3/2}, \quad (19)$$

we have calculated the values of the coefficient k corresponding to turbulence at a distance 40M from a grid with a mesh size M equal to 1", 3.25", and 5" using the empirical formula (17) in [4] and putting $b = \sqrt{\bar{u}^2}$ and $\bar{\epsilon} = (3/2)Ud\sqrt{\bar{u}^2}/dx$ in the notation of [4]:

M (inch)	1	3.25	5
k (cm ⁻¹)	0.197	0.065	0.042

With these values of k , the k curves in Figure 5 in [4], taking into account the correction for the wire length, are in good agreement with formula (19) for values of r not very large relative to L and

$$C = 3/2. \quad (20)$$

The local scale λ is so small that under the conditions of the experiments described in [4] the deviations from relations (18) for small r cannot be detected.²

² In this connection we note that the attempt of Millionshchikov [5] to apply the theory of isotropic turbulence neglecting third order moments to experimental data in [4] is based on a misunderstanding. It can easily be seen that under the conditions of the experiments in [4] the terms involving second order moments in the equations relating second order moments to third order ones (for example, in equation (3)) are *much less* than those involving third order moments.

In Figure 3 in [5], the comparison with the theoretical curve of R_{nn} (obtained by neglecting third order moments) using the experimental data in [4] was performed incorrectly since: 1) the data under consideration relate to a definite spectral component of fluctuations and not to the total fluctuations; and 2) the experimental data do not correspond to the condition $8\pi\epsilon t = 7.56$ which determines the theoretical curve.

In his later studies [6] (see also M.D. Millionshchikov, *Izv. Akad. Nauk SSSR Ser. Geogr. i Geofiz.* 5:4/5 (1941), 433-446 (in Russian)) Millionshchikov himself estimated the errors due to the neglect of third order moments, using subtle considerations involving third and fourth order moments.

The curves in Figure 28 in [4] cannot be used directly to determine C , since they do not include the correction for the wire length. However, for r small relative to L they confirm with sufficient accuracy the relation

$$(1 - R_{nn})/(1 - R_{dd}) = 4/3 \quad (21)$$

implied by (18). Including the correction for the wire length, the authors of [4] estimated the ratio on the left-hand side of (21) as 1.28 (see [4, p. 29]), which appears to be sufficiently close to the theoretical value $4/3$ if we take into account the restricted accuracy of an experiment.

References

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