

46. ON THE DEGENERATION OF ISOTROPIC TURBULENCE IN AN INCOMPRESSIBLE VISCOUS FLUID *

As in [1, 2], we assume that the velocity components

$$u_\alpha(P, t) = u_\alpha(x_1, x_2, x_3, t)$$

at the point $P = (x_1, x_2, x_3)$ at time t are random variables and denote by $\bar{A}$ the mathematical expectation of a random variable A .

In the case of isotropic turbulence in Taylor's sense (see [3, 4]) we have

$$\bar{u}_\alpha = 0, \quad (1)$$

and the second moments

$$b_{\alpha\beta} = \overline{u_\alpha(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, t) u_\beta(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, t)} \quad (2)$$

are given by the formulas (see [1, 5])

$$b_{\alpha\beta} = \bar{b}(r, t) \cos \theta_\alpha \cos \theta_\beta + \delta_{\alpha\beta} b_{nn}(r, t), \quad (3)$$

where

$$r^2 = (x_1^{(2)} - x_1^{(1)})^2 + (x_2^{(2)} - x_2^{(1)})^2 + (x_3^{(2)} - x_3^{(1)})^2,$$

$$x_\alpha^{(2)} - x_\alpha^{(1)} = r \cos \theta_\alpha, \quad \delta_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta, \quad \delta_{\alpha\beta} = 1 \text{ for } \alpha = \beta;$$

$$\bar{b}(r, t) = b_{dd}(r, t) - b_{nn}(r, t); \quad (4)$$

$$b_{dd}(r, t) = \overline{u_1(x_1, x_2, x_3, t) u_1(x_1 + r, x_2, x_3, t)}; \quad (5)$$

$$b_{nn}(r, t) = \overline{u_2(x_1, x_2, x_3, t) u_2(x_1 + r, x_2, x_3, t)}; \quad (6)$$

$$\partial b_{dd} / \partial r = -(2/r) \bar{b}. \quad (7)$$

Formula (7), together with

$$b_{dd}(0, t) = b_{nn}(0, t) = \overline{[u_\alpha(x_1, x_2, x_3, t)]^2} = b(t), \quad (8)$$

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makes it possible to express b_{nn} in terms of b_{dd} . Thus, all the second moments can be found from a single function $b_{dd}(r, t)$.

Von Karman and Howarth [5] derived the equation

$$\frac{\partial b_{dd}}{\partial t} + 2\left(\frac{\partial d_{nnd}}{\partial r} + \frac{4}{r}b_{nnd}\right) = 2\nu\left(\frac{\partial^2 b_{dd}}{\partial r^2} + \frac{4}{r}\frac{\partial b_{dd}}{\partial r}\right) \quad (9)$$

for $b_{dd}(r, t)$, where

$$b_{nnd}(r, t) = \overline{u_2^2(x_1, x_2, x_3, t)u_1(x_1 + r, x_2, x_3, t)}. \quad (10)$$

Loitsyanskii [6] showed that (9) implies that

$$\Lambda = \int_0^\infty b_{dd}(r, t)r^4 dr \quad (11)$$

does not depend on time.

As the "turbulence scale" we take the length

$$L = (\Lambda/b)^{1/5}. \quad (12)$$

Using L and the average value of the velocity components

$$\mathbf{u} = \sqrt{b}, \quad (13)$$

we introduce the Reynolds number

$$\text{Re} = L\mathbf{u}/\nu = (\Lambda^{1/5}/\nu)\mathbf{u}^{3/5}. \quad (14)$$

Since $\mathbf{u} \rightarrow 0$ as $t \rightarrow \infty$, (14) implies that the Reynolds number Re is small for large t . At this terminal stage of degeneration of isotropic turbulence it is legitimate to apply the theory neglecting the third moments. As was shown in [1], in this case

$$b_{dd}(r, t) = \frac{k}{(4\nu t)^{5/2}} \exp\left(-\frac{r^2}{8\nu t}\right), \quad (15)$$

$$b_{nn}(r, t) = \frac{k}{(4\nu t)^{5/2}} \left(1 - \frac{r^2}{8\nu t}\right) \exp\left(-\frac{r^2}{8\nu t}\right) \quad (16)$$

and consequently,

$$\mathbf{u} = At^{-5/4}, \quad (17)$$

$$L = B\sqrt{t}. \quad (18)$$

It is easy to express k , A and B in terms of ν and Λ .

For large Re , assuming that $b_{dd}(r, t)$ has the form

$$b_{dd}(r, t) = b\psi(r/L), \quad (19)$$

von Karman found (see [5]) that¹

$$\mathbf{u} = \mathbf{u}_0(t - t_0)^{-p}, \quad L = L_0(t - t_0)^{1-p}, \quad (20)$$

but he did not determine the exponent p .

Below we present a new derivation of formula (20), which implies that

$$p = 5/7. \quad (21)$$

We first show that for sufficiently large Re the assumption (19) implies

$$db/dt = -Kb^{3/2}L^{-1} \quad (22)$$

where K is an absolute constant. Formula (22) can be justified in different ways. If the assumption made in my paper [2] is adopted, then the following argument can be used. In isotropic turbulence the average energy per unit mass is equal to $(3/2)b$. Therefore the average energy dissipation rate per unit mass is

$$\bar{\epsilon} = -\frac{3}{2}db/dt. \quad (23)$$

Since using the notation $B_{dd}(r)$ in [2] we can write

$$b_{dd}(r) = b - \frac{1}{2}B_{dd}(r), \quad (24)$$

formula (23) in [2] implies that for r small relative to L and large relative to λ we have

$$b_{dd}(r) \approx b \left(1 - \frac{C}{2b} \bar{\epsilon}^{2/3} r^{2/3} \right). \quad (25)$$

For small ρ we derive the expression²

$$\psi(\rho) \approx 1 - g\rho^{2/3} \quad (26)$$

¹ Equation (77) in [6] was integrated incorrectly. Actually, the exponent in formula (78) must be equal to $AB/[5 + AB]$ and that in formula (81) must be equal to $-5/[5 + AB]$. The value $p = 5/7$ corresponds to $AB = 2$.

² We assume that relation (26) remains valid for arbitrarily small ρ . However, it has a real meaning only for ρ large relative to λ/L , since the assumption (19) itself can naturally be adopted only for r large relative to λ .

for the function $\psi(\rho)$, where

$$g = (C/2)\bar{\epsilon}^{2/3}L^{2/3}b^{-1}.$$

Consequently,

$$\bar{\epsilon} = (2gb/C)^{3/2}L^{-1}. \quad (27)$$

Putting

$$K = \frac{2}{3}(2g/C)^{3/2} \quad (28)$$

we obtain formula (22) from (23) and (27). Formula (28) shows how the coefficient K is expressed in terms of the absolute constant C introduced in [2] and the parameter g determined from (26) taking into account the form of the function $\psi(\rho)$.

From (12) and (22) we find that

$$db/dt = -K\Lambda^{-1/5}b^{17/10}. \quad (29)$$

Integration of equation (29) results in

$$b = (10/7K)^{10/7}\Lambda^{2/7}(t - t_0)^{-10/7}. \quad (30)$$

Relations (30), (13) and (12) imply

$$u = (10/7K)^{5/7}\Lambda^{1/7}(t - t_0)^{-5/7}, \quad (31)$$

$$L = (7K/10)^{2/7}\Lambda^{1/7}(t - t_0)^{2/7}. \quad (32)$$

A comparison of (31) and (32) with (20) shows that we actually have $p = 5/7$.

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References

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