

45. LOCAL STRUCTURE OF TURBULENCE IN AN INCOMPRESSIBLE VISCOUS FLUID AT VERY LARGE REYNOLDS NUMBERS *

§1. We denote by

$$u_\alpha(P) = u_\alpha(x_1, x_2, x_3, t), \quad \alpha = 1, 2, 3,$$

the velocity components at time t at a point with rectangular Cartesian coordinates x_1, x_2, x_3 . When studying turbulence it is natural to regard the velocity components $u_\alpha(P)$ at each point $P = (x_1, x_2, x_3, t)$ of the region G under consideration belonging to the four-dimensional space (x_1, x_2, x_3, t) as *random variables* in the sense of probability theory (for this approach, see the paper by Millionshchikov [1]).

Denoting the mathematical expectation of a random variable A by $\overline{A}$, we assume that

$$\overline{u_\alpha^2} \text{ and } \overline{(\partial u_\alpha / \partial x_\beta)^2}$$

are finite and bounded in each bounded subregion of the region G .

We introduce new coordinates in the four-dimensional space (x_1, x_2, x_3, t) :

$$y_\alpha = x_\alpha - x_\alpha^{(0)} - u_\alpha(P^{(0)})(t - t^{(0)}), \quad s = t - t^{(0)}, \quad (1)$$

where

$$P^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t^{(0)})$$

is a fixed point belonging to G . Note that the coordinates y_α of a point P depend on the random variables $u_\alpha(P^{(0)})$ and therefore they are themselves random variables. In the new coordinates the velocity components are

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)}). \quad (2)$$

Suppose that for fixed values of the components $u_\alpha(P^{(0)})$ the points $P^{(k)}$ ($k = 1, 2, \dots, n$) with coordinates $y_\alpha^{(k)}$ and $s^{(k)}$ in the coordinate system (1) lie in the region G . Then we can consider the $3n$ -dimensional conditional probability distribution F_n of the variables

$$w_\alpha^{(k)} = w_\alpha(P^{(k)}), \quad \alpha = 1, 2, 3; \quad k = 1, 2, \dots, n,$$

* Dokl. Akad. Nauk SSSR 30:4 (1941), 299-301.

for given

$$u_{\alpha}^{(0)} = u_{\alpha}(P^{(0)}).$$

In general, the distribution F_n depends on the parameters $x_{\alpha}^{(0)}, t^{(0)}, u_{\alpha}^{(0)}, y_{\alpha}^{(k)}$ and $s^{(k)}$.

Definition 1. A turbulence is said to be *locally homogeneous* in a region G if for any fixed n , $y_{\alpha}^{(k)}$ and $s^{(k)}$ the distribution F_n does not depend on $x_{\alpha}^{(0)}, t^{(0)}$, or $u_{\alpha}^{(0)}$ as long as all the points $P^{(k)}$ lie in G .

Definition 2. A turbulence is said to be *locally isotropic* in a region G if it is locally homogeneous and the distributions indicated in Definition 1 are invariant with respect to rotations and reflections of the original coordinate axes x_1, x_2, x_3 .

As compared to the notion of *isotropic turbulence* introduced by Taylor, the above definition of locally isotropic turbulence is *narrower* in the sense that it requires that the distributions F_n be independent of $t^{(0)}$, that is, stationary in time, and is *broaden* in the sense that the constraints are imposed only on the distributions of the velocity differences and not on the distribution of the velocities themselves.

§2. The isotropy hypothesis in Taylor's sense is in good agreement with experiment in the case of turbulence generated by a flow passing through a grid (see [3]). In the majority of other situations of practical significance, this hypothesis can only be regarded as a rather crude approximation to reality, even for small regions G and at very large Reynolds numbers.

By contrast, in our opinion, it seems very likely that in the case of an arbitrary turbulent flow with sufficiently large Reynolds number ¹

$$\text{Re} = LU/\nu$$

in sufficiently small regions G of the four-dimensional space (x_1, x_2, x_3, t) not lying close to the boundaries of the flow or other singularities of it, the hypothesis of local isotropy holds with high accuracy. Here, by "small regions" are meant those whose linear dimensions and time scales are small relative to L and

$$T = U/L,$$

¹ Here L and U are the typical length and velocity scales for the flow as a whole.

respectively.

Naturally, this hypothesis, which we have put forward in such a general and somewhat ambiguous form, cannot be proved rigorously.²

To make its experimental verification possible for individual special cases, we present below a number of consequences of the hypothesis of local isotropy.

§3. We denote by y a vector with components y_1, y_2, y_3 and consider the random variables

$$w_\alpha(y) = w_\alpha(y_1, y_2, y_3) = u_\alpha(x_1 + y_1, x_2 + y_2, x_3 + y_3, t) - u_\alpha(x_1, x_2, x_3, t). \quad (3)$$

By virtue of the hypothesis of local isotropy, the distributions of (3) do not depend on x_1, x_2, x_3 or t . For the first moments of the variables $w_\alpha(y)$,

² Here we present only some general consideration in favour of the suggested hypothesis. For very large Re a turbulent flow can be represented in the following way. The mean flow (characterized by the mathematical expectations $\bar{u}_\alpha$) is accompanied by "first-order fluctuations" imposed on it, consisting of chaotic motions of individual fluid volumes with diameters of the order of $l^{(1)} = l$ (where l is the Prandtl mixing length) relative to one another; we denote by $v^{(1)}$ the order of the velocities of these relative motions. In turn, the first-order fluctuations become unstable at very large Re and there appear "second-order fluctuations" imposed on the former, the latter having mixing length $l^{(2)} < l^{(1)}$ and relative velocities $v^{(2)} < v^{(1)}$. This process of successive refinement of turbulent fluctuations goes on until for fluctuations of sufficiently high order n the Reynolds number $Re^{(n)} = l^{(n)}v^{(n)}/\nu$ turns out to be sufficiently small, so that the effect of viscosity on the n th-order fluctuations is appreciable and prevents the formation of $(N+1)$ th-order fluctuations imposed on the former.

From the energetic standpoint, the process of turbulent mixing can be thought of in the following manner: the first-order fluctuations absorb the energy of the mean motion and transfer it subsequently to higher-order fluctuations; as to the energy of the smallest fluctuations, it is dissipated and transformed into thermal energy due to viscosity.

By virtue of the chaotic mechanism of transfer of motion from lower-order fluctuations to higher-order ones, it is natural to assume that, within the limits of space regions which are small as compared to $l^{(1)}$, small-scale higher-order fluctuations are subject to an approximately spatially isotropic statistical regime. Within short time intervals this regime can naturally be regarded as being stationary, even when the flow as a whole is non-stationary.

Since at very large Re the velocity component differences

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)})$$

at nearby points P and $P^{(0)}$ of the four-dimensional space (x_1, x_2, x_3, t) are determined almost exclusively by higher-order fluctuations, the presented scheme leads to the hypothesis of local isotropy in small regions in the sense of the definitions in §§1, 2.

local isotropy implies that

$$\overline{w_\alpha(y)} = 0. \quad (4)$$

Therefore we proceed to the study of the second moments³

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \overline{w_\alpha(y^{(1)})w_\beta(y^{(2)})}. \quad (5)$$

Local isotropy implies that

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \frac{1}{2}[B_{\alpha\beta}(y^{(1)}, y^{(1)}) + B_{\alpha\beta}(y^{(2)}, y^{(2)}) - B_{\alpha\beta}(y^{(2)} - y^{(1)}, y^{(2)} - y^{(1)})] \quad (6)$$

This formula allows us to confine ourselves to second moments of the form $B_\alpha(y, y)$. For these we have

$$B_{\alpha\beta}(y, y) = \overline{B}(r) \cos \theta_\alpha \cos \theta_\beta + \delta_{\alpha\beta} B_{nn}(r), \quad (7)$$

where $r^2 = y_1^2 + y_2^2 + y_3^2$, $y_\alpha = r \cos \theta_\alpha$, $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$;

$$\overline{B}(r) = B_{dd}(r) - B_{nn}(r), \quad (8)$$

$$B_{dd}(r) = \overline{[w_1(r, 0, 0)]^2}, \quad B_{nn}(r) = \overline{[w_2(r, 0, 0)]^2}. \quad (9)$$

For $r = 0$ we have

$$B_{dd}(0) = B_{nn}(0) = \frac{\partial}{\partial r} B_{dd}(0) = \frac{\partial}{\partial r} B_{nn}(0) = 0, \quad (10)$$

$$\frac{\partial^2}{\partial r^2} B_{dd}(0) = 2 \overline{\left(\frac{\partial w_1}{\partial y_1}\right)^2} = 2a^2, \quad \frac{\partial^2}{\partial r^2} B_{nn}(0) = 2 \overline{\left(\frac{\partial w_2}{\partial y_1}\right)^2} = 2a_n^2. \quad (11)$$

Formulas (6)–(11) have been derived without using the assumption that the fluid is incompressible. This assumption implies the equation

$$r \partial B_{dd} / \partial r = -2\overline{B}, \quad (12)$$

making it possible to express B_{nn} in terms of B_{dd} . Formulas (12) and (11) imply that

$$a_n^2 = 2a^2. \quad (13)$$

³ All results in §3 are completely analogous to those obtained in [1, 2, 4] for the case of isotropic turbulence in Taylor's sense.

Further, it is easy to show that (under the assumption of incompressibility) the average energy dissipation rate per unit mass is given by

$$\bar{\epsilon} = \nu \left\{ 2 \overline{\left(\frac{\partial w_1}{\partial y_1} \right)^2} + 2 \overline{\left(\frac{\partial w_2}{\partial y_2} \right)^2} + 2 \overline{\left(\frac{\partial w_3}{\partial y_3} \right)^2} + \overline{\left(\frac{\partial w_2}{\partial y_1} + \frac{\partial w_1}{\partial y_2} \right)^2} + \overline{\left(\frac{\partial w_3}{\partial y_2} + \frac{\partial w_2}{\partial y_3} \right)^2} + \overline{\left(\frac{\partial w_1}{\partial y_3} + \frac{\partial w_3}{\partial y_1} \right)^2} \right\} = 15\nu a^2. \quad (14)$$

§4. Consider the transformation of coordinates

$$y'_\alpha = y_\alpha / \eta, \quad s' = s / \sigma. \quad (15)$$

In the new coordinates the velocities, the kinetic viscosity, and the average energy dissipation rate per unit mass are expressed by the following formulas:

$$w'_\alpha = w_\alpha \sigma / \eta, \quad \nu' = \nu \sigma / \eta^2, \quad \bar{\epsilon}' = \bar{\epsilon} \sigma^3 / \eta^2. \quad (16)$$

We now state the following hypothesis.

The first similarity hypothesis. *The distributions F_n for locally isotropic turbulence are uniquely determined by the parameters ν and $\bar{\epsilon}$.*

For

$$\eta = \lambda = \sqrt{\nu/a} = \nu^{3/4} / \bar{\epsilon}^{1/4}, \quad (17)$$

$$\sigma = 1/a = \sqrt{\nu/\bar{\epsilon}} \quad (18)$$

the transformation of coordinates (15) results in the parameters $\nu' = 1$, $\bar{\epsilon}' = 1$.

Therefore by virtue of the above similarity hypothesis, the corresponding function

$$B'_{dd}(r') = \beta_{dd}(r') \quad (19)$$

must be the same in all cases of locally isotropic turbulence. The formula

$$B_{dd}(r) = \sqrt{\nu \bar{\epsilon}} \beta_{dd}(r/\lambda) \quad (20)$$

in combination with the earlier derived expressions shows that in the case of locally isotropic turbulence the second moments $B_{\alpha\beta}(y^{(1)}, y^{(2)})$ can be uniquely expressed in terms of ν , $\bar{\epsilon}$ and the universal function β_{dd} .

§5. To determine the behaviour of the function $\beta_{dd}(r')$ for large r' we introduce one more hypothesis.

The second similarity hypothesis.⁴ *If the absolute values of the vectors $y^{(k)}$ and their differences $y^{(k)} - y^{(k')}$ (where $k' \neq k$) are large relative to λ , the distributions F_n are uniquely determined by the parameter $\bar{\epsilon}$ and do not depend on ν .*

We put

$$y''_{\alpha} = y'_{\alpha}/k^3, \quad s'' = s'/k^2, \quad (21)$$

where y'_{α} and s' are determined according to formulas (15), (17) and (18). Since

$$\bar{\epsilon}' = \bar{\epsilon}'' = 1,$$

for any k , by virtue of the above hypothesis, for r' large relative to $\lambda' = 1$ we have

$$B''_{dd}(r'') \approx B'_{dd}(r'') = \beta_{dd}(r'/k^3).$$

On the other hand, formula (20) implies that

$$B''_{dd}(r'') = (1/k^2)B'_{dd}(r') = (1/k^2)\beta_{dd}(r').$$

Hence, for large r' we have

$$\beta_{dd}(r'/k^3) \approx (1/k^2)\beta_{dd}(r'),$$

whence

$$\beta_{dd}(r') \approx C(r')^{2/3}, \quad (22)$$

where C is an absolute constant. By virtue of (17), (20) and (22) for r large relative to λ we have

$$B_{dd}(r) \approx C\bar{\epsilon}^{2/3}r^{2/3}. \quad (23)$$

From (23) and (12) it can easily be derived that for r large relative to λ ,

$$B_{nn}(r) \approx 4/3B_{dd}(r). \quad (24)$$

⁴ In terms of the schematic description of turbulence presented in Footnote 2, λ is the scale of the smallest fluctuations, whose energy is dissipated and transformed directly into thermal energy due to viscosity. The meaning of the second similarity hypothesis is that for fluctuations of intermediate orders for which $l^{(k)}$ is greater than λ the mechanism of energy transfer from larger fluctuations to smaller ones does not depend on the viscosity.

We note that instead of formula (24), by virtue of (13) for r small as compared to λ the relation

$$B_{nn}(r) \approx 2B_{dd}(r) \quad (25)$$

holds.

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References

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